A Nonlinear, Fully Coupled Moving Mesh Adaptive Grid Strategy

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In the numerical simulation of complex physical phenomena, the crucial requirement is predictability, i.e., that the simulation results remain faithful to the actual physical processes. Accordingly, the generation and accumulation of numerical error during the simulation is of special concern, since it introduces distortions that fundamentally alter the fidelity of the simulation. Errors resulting from a lack of spatial resolution are particularly deleterious. However, over-resolving is computationally expensive.

Adaptive grids attempt to provide sufficient resolution where needed while minimizing the computational cost of the simulation. Our emphasis is on moving grid methods (also known as *r*-refinement), where grid points are able to move to follow the solution. The grid positions are determined from a suitable grid evolution equation. While many grid evolution equations have been proposed in the literature [1], here we focus on harmonic maps [2], which are desirable because, under certain conditions, they guarantee the existence and uniqueness of the grid mapping. In two dimensions (2-D), for a scalar error monitor function $w(\vec{x},t)$, harmonic function theory results in Winslow's variable diffusion method grid evolution equation for $\xi(\vec{x})$,$\nabla\cdot(\frac{1}{\omega}\nabla\xi)=0$, with ξ the logical variable and x spanning the configuration space. The inverse of this equation [to determine $\vec{x}(\xi)$] can be readily expressed in terms of the contravariant metric tensor components $g^{ij}=\nabla\xi i\cdot\nabla\xi j$ and the Jacobian of the transformation $\vec{x}(\xi)$, J, as:

$$E(w,\vec{x})=\frac{\partial}{\partial\xi_i}\left(\frac{Jg^{ij}}{w}\right)=0\ ;\ j=1,2 \qquad (1)$$

This is the (nonlinear) grid evolution equation for a given error function w (which generally depends on the solution of a physical model) [3, 4]. For time-stepping problems, it is common to relax the elliptic constraint in Eq. 1 by introducing a time-dependent term. This is advantageous for numerical stability, as perturbations in the solution of Eq. 1 may grow undamped [5] to the point of folding the grid. Here, we employ the following choice:

$$E(\bar{w},\mathbf{x}^{n+1})=e^{-\Delta t/\tau}E^n(\bar{w},\mathbf{x}^n)\ ;$$

$$\bar{w}=0.5(w^{n+1}+w^n), \qquad (2)$$

with τ an adjustable parameter, which in general we take proportional to the time step. It can be shown [5] that this choice appropriately damps arising perturbations while asymptotically satisfying Eq. 1 for t » τ.

One drawback of harmonic function theory is that the resulting grid evolution equation is generally very nonlinear and stiff.

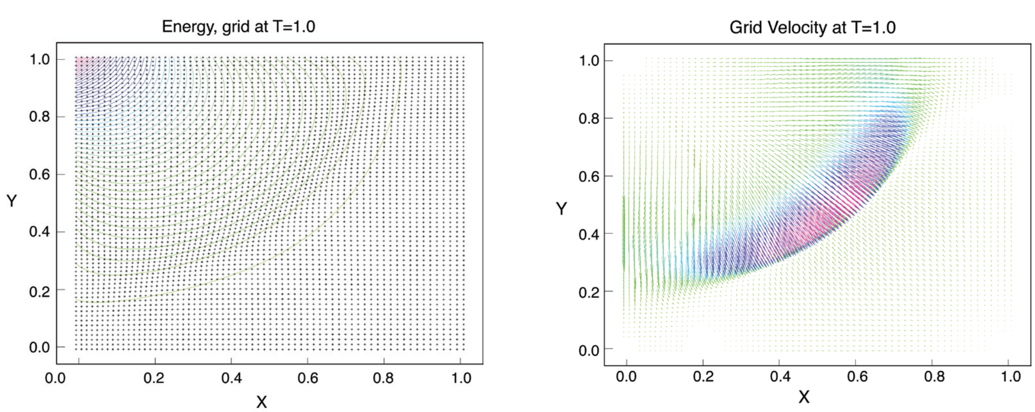

***Fig. 1.* Snapshot of implicit moving mesh computation in a 128 x 128 grid.**

Table 1. Grid convergence study for $\Delta t = 0.1$, $\tau = 2\Delta t$, reporting nonlinear (NLI), linear (GM) iterations per time step.

Grid	NLI/Δt	GM/Δt	CPU (s)
16x16	6	1	22
32x32	6	1	89
64x64	6	1.3	350
128x128	6	1.5	1396

Furthermore, physics models for which spatial adaptation is necessary are typically very stiff as well. For such systems, implicit temporal schemes are preferred for efficiency, as they allow one to use time steps comparable to the dynamical time scale of interest in the problem at hand. However, when coupled to a grid evolution equation, such large implicit time steps may not be advantageous from an accuracy standpoint unless both grid and physics equations are solved in coupled manner.

The coupled nonlinear solution of such physics-grid systems represents, however, a formidable numerical challenge. It is this challenge that we undertake in this research. At the heart of the matter is to demonstrate that developing a scalable, efficient nonlinear algorithm to solve such systems is indeed possible. We base our strategy on Newton-Krylov methods [6], which are ideally suited for this task owing to their robustness and the possibility of preconditioning. We employ a lagged-grid strategy for preconditioning, where the current grid generation step is based on the physics field at the previous Newton step. This is equivalent to the standard practice in the moving mesh community of splitting the grid generation step from the evolution of the physical system. Because this approach effectively decouples the grid and the physics in the preconditioner, it allows one to straightforwardly transfer successful preconditioning technologies in fixed grids to the moving mesh framework.

Such preconditioning approach requires an effective standalone solver for Eq. 2. This has been recently proposed in Ref. [7]. Here, we report on recent results that demonstrate that a scalable nonlinear solver for the coupled nonlinear physics-grid system in two dimensions is indeed possible. We focus on the equilibrium radiation-diffusion model in the matter-dominated regime as the physical system of interest, which is described by:

$$\partial_t T = \nabla \cdot (D_L \nabla E) \, ;$$

$$D_L^{-1} = \frac{3}{T^3} + \left| \frac{\nabla E}{E} \right| \; ; \; E = T^4,$$

where D_L is a limited nonlinear diffusion coefficient. We use a gradient-based error estimator,

$$w = \sqrt{1 + (\nabla T)^2}$$

as the monitor function in Eq. 2. We initialize the calculation with a bilinear profile in the temperature, with corner values in a 2-D domain given by: $T(0, 1) = 1.0; T(1, 1) = T(1, 0) = T(0, 0) = 0.2$.

The nonlinear evolution of this system results in an oblique front propagating from the top-left corner to bottom-right corner. A snapshot is shown in Fig. 1 (left). The corresponding grid velocity is shown in Fig. 1 (right), demonstrating that the nodes are indeed following the front. The grid convergence study in Table 1 demonstrates that the nonlinear solver scales optimally with grid refinement, with CPU proportional to the grid size. Future work will consider multimaterial configurations, the use of more rigorous error estimators, and the cost-effectiveness of this approach vs fixed grids.

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